

Chapter 2: Transformations and Expectations

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Chapter 2: roadmap

- 2.1 Distributions of Functions of a Random Variable
- 2.2 Expected Values
- 2.3 Moments and Moment Generating Functions
- 2.4 Differentiating under an integral or sum (Not covered)

Key idea

A transformation changes the distribution of a random variable; expectation and moments summarize important features of the resulting distribution.

Distributions of Functions of a Random Variable

Textbook: Section 2.1

2.1 Distributions of Functions of a Random Variable

Let X be a random variable and let

$$Y = g(X).$$

Then the probability law of Y is induced by the law of X through the function g .

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(X \in g^{-1}(A)),$$

where

$$g^{-1}(A) = \{x : g(x) \in A\}.$$

Key idea

To find the distribution of Y , translate an event about Y back into an event about X .

Discrete transformations

If X is discrete and $Y = g(X)$, then Y is also discrete. Its pmf is

$$f_Y(y) = \mathbb{P}(Y = y) = \sum_{x:g(x)=y} f_X(x).$$

Thus values of X that map to the same y must have their probabilities added.

Example

If X takes values $-1, 0, 1$ with probabilities $1/4, 1/2, 1/4$ and $Y = X^2$, then

$$\mathbb{P}(Y = 0) = \frac{1}{2}, \quad \mathbb{P}(Y = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Discrete transformations

Example: Binomial transformation

Suppose

$$X \sim \text{Binomial}(n, p), \quad Y = n - X.$$

For $y = 0, 1, \dots, n$, the equation $n - x = y$ gives $x = n - y$. Hence

$$f_Y(y) = f_X(n - y) = \binom{n}{n - y} p^{n - y} (1 - p)^y = \binom{n}{y} (1 - p)^y p^{n - y}.$$

Therefore

$$Y \sim \text{Binomial}(n, 1 - p).$$

Key idea

The transformation keeps the binomial family but replaces p by $1 - p$.

Continuous transformations – general approach

CDF method: the most general approach

For any transformation $Y = g(X)$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y).$$

The CDF method is especially useful when:

- g is not one-to-one;
- the inverse has several branches;
- the support changes in a nontrivial way.

Once F_Y is found, differentiate when appropriate:

$$f_Y(y) = F'_Y(y).$$

Continuous transformations – general approach

Definition

For a set A in the range of g , the **inverse image** is

$$g^{-1}(A) = \{x : g(x) \in A\}.$$

If g is one-to-one, then for a point y we can write the ordinary inverse function $x = g^{-1}(y)$.

If g is not one-to-one, $g^{-1}(\{y\})$ can contain several points.

Example: Square transformation

For $g(x) = x^2$,

$$g^{-1}(\{y\}) = \{-\sqrt{y}, \sqrt{y}\}, \quad y > 0.$$

This is why a non-monotone transformation generally requires summing contributions from multiple inverse branches.

Continuous transformations – general approach

Example: Square transformation continued

For $Y = X^2$ and $y > 0$,

$$F_Y(y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

When X has a differentiable cdf,

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, \quad y > 0.$$

Key idea

Both roots $+\sqrt{y}$ and $-\sqrt{y}$ contribute to the density of Y .

Continuous transformations – general approach

Example: Square transformation continued

Suppose that the continuous random variable X has pdf

$$f_X(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

Find the pdf of $Y = X^2$..

Continuous transformations – general approach

For $0 < y < 1/4$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x) dx = 2\sqrt{y}.$$

Therefore,

$$F_Y(y) = \begin{cases} 0, & y \leq 0, \\ 2\sqrt{y}, & 0 < y < \frac{1}{4}, \\ 1, & y \geq \frac{1}{4}. \end{cases}$$

Differentiating,

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{y}}, & 0 < y < \frac{1}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Continuous transformations – general approach

Example: Uniform transformation

Let

$$X \sim U(0, 2\pi), \quad Y = \sin^2 X.$$

Then $0 \leq Y \leq 1$. For $0 < y < 1$, the event

$$\{\sin^2 X \leq y\}$$

consists of several intervals because $\sin^2 x$ is not monotone on $(0, 2\pi)$.

Key idea

The transformation is simple, but its inverse image is not. Draw $y = \sin^2 x$ and identify all intervals satisfying the inequality before computing probabilities.

Example 2.1.2: locating the inverse branches

Let

$$a = \arcsin(\sqrt{y}), \quad 0 < a < \frac{\pi}{2}.$$

Then $\sin^2 x \leq y$ on intervals of total length $4a$ inside $(0, 2\pi)$, so

$$F_Y(y) = \frac{4a}{2\pi} = \frac{2}{\pi} \arcsin(\sqrt{y}), \quad 0 < y < 1.$$

Differentiating gives

$$f_Y(y) = \frac{1}{\pi \sqrt{y(1-y)}}, \quad 0 < y < 1.$$

Continuous transformations – general approach

Always identify the support first

Define

$$\mathcal{X} = \{x : f_X(x) > 0\}, \quad \mathcal{Y} = \{g(x) : x \in \mathcal{X}\}.$$

Before differentiating or applying a transformation formula:

1. determine $\mathcal{X}$;
2. map $\mathcal{X}$ through g to obtain $\mathcal{Y}$;
3. determine whether g is monotone on $\mathcal{X}$;
4. if not, partition $\mathcal{X}$ into monotone pieces.

Key idea

Many transformation mistakes are support mistakes rather than calculus mistakes.

Continuous transformations– monotone case

Theorem (Monotone transformations of CDFs)

Let $Y = g(X)$ and suppose g is monotone on the support $\mathcal{X}$.

1. If g is increasing,

$$F_Y(y) = F_X(g^{-1}(y)), \quad y \in \mathcal{Y}.$$

2. If g is decreasing and X is continuous,

$$F_Y(y) = 1 - F_X(g^{-1}(y)), \quad y \in \mathcal{Y}.$$

Proof idea

Translate $\{g(X) \leq y\}$ into an inequality involving X . An increasing g preserves the inequality direction; a decreasing g reverses it.

Continuous transformations– monotone case

Example: Uniform-exponential relationship

Let $X \sim U(0, 1)$ and define

$$Y = -\log X.$$

The transformation is decreasing and

$$g^{-1}(y) = e^{-y}, \quad y > 0.$$

Therefore

$$F_Y(y) = 1 - F_X(e^{-y}) = 1 - e^{-y}, \quad y > 0.$$

Hence

$$f_Y(y) = e^{-y}, \quad y > 0,$$

so Y is exponential with mean 1.

Continuous transformations– monotone case

Theorem (monotone density transformation)

Suppose X has continuous pdf f_X on its support $\mathcal{X}$, $Y = g(X)$, and g is monotone on $\mathcal{X}$. If g^{-1} is differentiable on $\mathcal{Y}$, then

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & y \in \mathcal{Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose g is increasing. Then

$$F_Y(y) = F_X(g^{-1}(y)).$$

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

If g is decreasing, the derivative of the inverse is negative; the absolute value makes the density nonnegative.

Continuous transformations– monotone case

Example: Inverted gamma pdf

Suppose X has a gamma-type density on $(0, \infty)$,

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0,$$

and let

$$Y = \frac{1}{X}.$$

Then $g^{-1}(y) = 1/y$ and

$$\left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{y^2}.$$

Thus

$$f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{-(\alpha+1)} e^{-1/(\beta y)}, \quad y > 0.$$

A transformation checklist

For $Y = g(X)$:

1. Write the support $\mathcal{X}$ of X .
2. Find the range $\mathcal{Y} = g(\mathcal{X})$.
3. Check whether g is one-to-one on $\mathcal{X}$.
4. Solve $y = g(x)$ for the inverse branch(es).
5. Apply the CDF method or the density transformation theorem.
6. State the answer with its support.
7. Check that the resulting pdf integrates to 1.

Continuous transformations – piecewise monotone case

Theorem (Piecewise monotone transformation)

Suppose the support of X can be partitioned into sets $A_1, \dots, A_k$ (apart from a probability-zero set) such that on each A_i the transformation g_i is monotone and differentiable through its inverse. If every branch maps onto the same relevant range $\mathcal{Y}$, then

$$f_Y(y) = \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right|, \quad y \in \mathcal{Y}.$$

Key idea

The density at y is the sum of the probability contributions from every inverse branch leading to y .

Continuous transformations – piecewise monotone case

Example: Normal-chi-squared relationship

Let $X \sim N(0, 1)$ and $Y = X^2$. The two inverse branches are

$$x_1(y) = -\sqrt{y}, \quad x_2(y) = \sqrt{y}, \quad y > 0.$$

Using Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \phi(-\sqrt{y}) \frac{1}{2\sqrt{y}} + \phi(\sqrt{y}) \frac{1}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi y}} e^{-y/2}, \quad y > 0. \end{aligned}$$

This is the χ^2 density with one degree of freedom.

Probability integral transformation

The generalized inverse of a CDF

For a nondecreasing cdf F , define its generalized inverse by

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad 0 < u < 1.$$

This definition remains single-valued even when F has flat portions.

Key idea

The generalized inverse is the mathematical basis for quantiles and inverse-CDF simulation.

Probability integral transformation

Theorem (Probability integral transformation)

If X has a continuous cdf F_X , then

$$U = F_X(X) \sim U(0, 1).$$

Equivalently,

$$\mathbb{P}(U \leq u) = u, \quad 0 < u < 1.$$

Proof idea

Using the generalized inverse,

$$\mathbb{P}(F_X(X) \leq u) = \mathbb{P}(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u.$$

Probability integral transformation

The probability integral transform can be reversed. If $U \sim U(0, 1)$ and F is a continuous cdf, then

$$X = F^{-1}(U)$$

has cdf F .

Example

For an exponential distribution with cdf

$$F(x) = 1 - e^{-x/\lambda}, \quad x \geq 0,$$

solve $u = 1 - e^{-x/\lambda}$:

$$X = -\lambda \log(1 - U).$$

Thus uniform random numbers can generate exponential random variables.

Quiz 1: transformations

Quiz

1. If $X \sim U(0, 1)$ and $Y = X^2$, find the support and pdf of Y .
2. If X is discrete with $\mathbb{P}(X = -1) = 1/4$, $\mathbb{P}(X = 0) = 1/2$, $\mathbb{P}(X = 1) = 1/4$, find the pmf of $Y = |X|$.
3. Let $X \sim N(0, 1)$ and $Y = |X|$.
 - a What is the support of Y ?
 - b Write $F_Y(y)$ for $y \geq 0$ in terms of the standard normal cdf Φ .
 - c Find $f_Y(y)$.