

Probability Theory

Chapter 1—Combinatorial Analysis

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¹These slides are based on Prof. Mui Li's. Any mistakes are my own. If you find an error, please send an email at nanliu@xmu.edu.cn

Introduction to Probability

- Probability theory yields mathematical tools to deal with uncertain events.
- Applications include
 - Population genetics: tree-valued stochastic processes,
 - Web search engine: Markov chain theory,
 - Data mining, Machine learning: Stochastic gradient, Markov chain Monte Carlo,
 - Image processing: Markov random fields,
 - Design of wireless communication systems: random matrix theory,
 - Optimization of engineering processes: simulated annealing, genetic algorithms,
 - Finance : option pricing, volatility models.

1.1 Combinatorial analyses aka Counting

- Many **basic** probability problems are counting problems.
- *Example:* Assume there are 1 man and 2 women in a room. You pick a person randomly. What is the probability P_1 that this is a man? If you pick two persons randomly, what is the probability P_2 that these are a man and woman
- *Answer:* You have the possible outcomes: (M), (W1), (W2) so

$$P_1 = \frac{\# \text{ "successful" events}}{\# \text{ events}} = \frac{\# \text{ boys}}{\# \text{ boys} + \# \text{ girls}} = \frac{1}{3}.$$

To compute P_2 , you can think of all the possible events: (M,W1), (M,W2), (W1,W2) so

$$P_2 = \frac{\# \text{ "successful" events}}{\# \text{ events}} = \frac{2}{3}.$$

- Both problems consists of counting the number of different ways that a certain event can occur.

1.2 Basic Principle of Counting

- **Basic Principle of Counting:** Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of n_1 possible outcomes and if, for each outcome of experiment 1, there are n_2 possible outcomes of experiment 2, then there are $n_1 \times n_2$ possible outcomes of the two experiments.
- *Example:* A football tournament consists of 14 teams, each of which has 11 players. If one team and one of its players are to be selected as team and player of the year, how many different choices are possible?
- *Answer:* Selecting the team can be regarded as the outcome of the first experiment and the subsequent choice of one of its player as the outcome of the second experiment, so there are $14 \times 11 = 154$ possibilities.

1.2 Generalized Principle of Counting

- **Generalized Principle of Counting:** If r experiments that are to be performed are such that the 1st one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the 2nd experiment; and if, for each of the $n_1 \times n_2$ possible outcomes of the first two experiments, there are n_3 possible outcomes of the 3rd experiment; and if..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of the r experiments.
- *Example:* A university committee consists of 4 undergrads, 5 grads, 7 profs and 2 non-university persons. A subcommittee of 4, consisting of 1 person from each category, is to be chosen. How many different subcommittees are possible?
- *Answer:* The choice of a subcommittee is the combined outcome of the four separate experiments of choosing a single representative from each of the categories. So it follows that, there are

$$4 \times 5 \times 7 \times 2 = 280$$

possible subcommittees.

1.2 Generalized Principle of Counting: More Examples

- *Example:* How many different 6-place license plates are possible if the first 3 places are to be occupied by letters and the final 3 by numbers?
- *Answer:* We have simply

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,760,000.$$

- *Example:* How many functions defined on n points are possible if each functional value is either 0 or 1?
- *Answer:* We have

$$f = (f(1), f(2), \dots, f(n))$$

where $f(i) \in \{0, 1\}$ so there are

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

possibilities.

1.2 Generalized Principle of Counting: More Examples

- *Example:* How many different 6-place license plates are possible if the first 3 places are to be occupied by letters, the final 3 by numbers and if repetition among letters were prohibited, repetition among numbers were prohibited, repetition among both letters and numbers were prohibited?
- *Answer:* We have

letters prohib.	$26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15,600,000$
numbers prohib.	$26 \times 26 \times 26 \times 10 \times 9 \times 8 = 12,654,720$
letters & numbers prohib.	$26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$

1.3 Permutations

- *Example:* Consider the acronym XMU. How many different ordered arrangements of the letters X, M and U are possible?
- *Answer:* We have (X,M,U), (X,U,M), (M,X,U), (U,X,M), (M,U,X) and (U,M,X); i.e. 6 possible arrangement. Each arrangement is known as a *permutation*.
- **General Result.** Suppose you have n objects. The number of permutations of these n objects is given by

$$n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

Remember the convention $0! = 1$.

- *Example:* Assume we have an horse race with 12 horses. How many possible rankings are (theoretically) possible?

$$12! = 479,001,600$$

1.3 Permutations: Examples

- *Example:* A class in “Probability Theory” consists of 15 men and 20 women. An examination is given and the students are ranked according to their performance. Assume that no two students obtain the same score.
 - How many different rankings are possible?
 - If the men are ranked among themselves and the women among themselves, how many different rankings are possible?
- *Solution:*
 - Each ranking corresponds to a particular ordered arrangement of the $15+20=35$ people. So there are $35!$ possible rankings.
 - Number of ranking for men is $15!$ and for women $20!$

1.3 Permutations: More Examples

- *Example:* You have 10 textbooks that you want to order on your bookshelf: 3 mathematics books, 3 physics books, 2 chemistry books and 2 biology books. You want to arrange them so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?
- *Solution:* For each ordering of the subject, say M/P/C/B or P/B/C/M, there are

$$3!3!2!2! = 144$$

arrangements. As there are $4!$ ordering of the subjects, then you have

$$144 \cdot 4! = 3456$$

possible arrangements.

1.3 Permutations: More Examples

- *Example:* How many different letter arrangements can be formed from the letters EEP₁P₂P₃R?
- *Solution:* There are 6! possible permutations of letters E₁E₂P₁P₂P₃R but the letters are not labelled so we cannot distinguish E₁ and E₂ and P₁, P₂ and P₃; e.g. E₁P₁E₂P₂P₃R cannot be distinguished from E₂P₁E₁P₂P₃R and E₂P₂E₁P₁P₃R. That is if we permuted the E's and the P's among themselves then we still have EEP₁P₂P₃R. We have 2!3! permutations of the labelled letters of the form EEP₁P₂P₃R. Hence there are

$$\frac{6!}{2!3!} = 60$$

possible arrangements of the letters EEP₁P₂P₃R.

- **General Result.** Suppose you have n objects. The number of different permutations of these n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike is given by

$$\frac{n!}{n_1!n_2! \cdots n_r!} := \binom{n}{n_1, n_2, \dots, n_r}.$$

1.3 Permutations: More Examples

- *Example:* A speed skating tournament has 4 competitors from South Korea, 3 from Canada, 3 from China, 2 from the USA and 1 from France. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?
- *Answer:* We simply have

$$\frac{(4 + 3 + 3 + 2 + 1)!}{4!3!3!2!1!} = 3,603,600.$$

1.4 Combinations

- We want to determine the number of different groups of r objects that could be formed from a total of n objects.
- *Example:* How many different groups of 3 could be selected from A,B,C,D and E?
- *Answer:* There are 5 ways to select the 1st letter, 4 to select the 2nd and 3 to select the 3rd so $5 \cdot 4 \cdot 3 = 60$ ways to select WHEN the order in which the items are selected is relevant. When it is not relevant, then say the group BCE is the same as BEC, CEB, CBE, EBC, ECB; there are $3! = 6$ permutations. So when the order is irrelevant, we have $60/6 = 10$ different possible groups.
- **General result:** When the order of selection is relevant, there are

$$n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!} \quad \text{possible groups.}$$

When the order of selection is irrelevant, there are

$$\frac{n!}{(n-r)!r!} := \binom{n}{r} \quad \text{possible groups.}$$

1.4 Combinations: Examples

- *Example:* From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What is the probability that 2 of the men are feuding and refuse to be served on the committee together?
- *Answer:* We have $\binom{5}{2} = 10$ possible W groups and $\binom{7}{3} = 35$ possible M groups, so $10 \cdot 35 = 350$ groups. In the 35 groups, we have $5 = \binom{2}{2} \binom{5}{1}$ groups where the 2 feuding men can be, so there are $10 \cdot 30 = 300$ possible committees.

1.4 Combinations: More Examples

- *Example:* Assume we have a set of n antennas of which m are defective. All the defective and all the functionals are indistinguishable. How many linear orderings are there in which no two defectives are consecutive?
- *Answer:* If two defective antennas cannot be consecutive, then the space among the $n - m$ functional antennas can contain at most one defective antenna. There are $n - m + 1$ positions possible and select m of them to put the defective antennas, so that is

$$\binom{n - m + 1}{m}$$

configurations which is obviously equal to zero if $m > n - m + 1$.

Recapitulation

- **Principle of counting:** If experiment 1 has n_1 possible outcomes, experiment 2 has n_2 possible outcomes, ..., experiment r has n_r possible outcomes, then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of the r experiments.
- **Permutations:** For n objects, the number of permutations, i.e. ordered arrangements, of these n objects is given by $n!$
- **Permutations with objects alike:** For n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike, there are

$$\frac{n!}{n_1! n_2! \dots n_r!}$$

permutations.

- **Combinations:** The number of different groups of r objects that can be formed from n objects is

$$\begin{array}{ll} \frac{n!}{(n-r)!} & \text{if the order matters,} \\ \frac{n!}{(n-r)! r!} & \text{if the order does not matter.} \end{array}$$

- **Question** (*Pb. 13*): Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?
- **Question** (*Pb. 15*): A dance class consists of 22 students, 10 W and 12 M. If 5 M and 5 W are to be chosen and then paired off, how many results are possible?
- **Question** (*Pb. 20*): A person has 8 friends, of whom 5 will be invited to a party.
 - (a) How many choices are there if 2 of the friends are feuding and will not attend together?
 - (b) How many choices if 2 of the friends will only attend together?

1.4 Continue: The Binomial Theorem

Theorem

We have

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

- **Example:** Compute $(x + y)^3$.
- **Answer:** We have

$$\begin{aligned}(x + y)^3 &= \binom{3}{0} x^0 y^3 + \binom{3}{1} x y^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3 y^0 \\ &= y^3 + 3xy^2 + 3x^2y + x^3.\end{aligned}$$

Method 1: Proof of the binomial theorem by induction

This is true at rank $n = 1$. Assume this is true at rank $n - 1$ and let us prove it at rank n . We have

$$\begin{aligned}(x + y)^n &= (x + y)(x + y)^{n-1} = (x + y) \left(\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} \right) \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}.\end{aligned}$$

For the 1st term on the lhs, perform a change of variables $l \leftarrow k + 1$ then we have

$$(x + y)^n = x^n + \sum_{l=1}^{n-1} \left\{ \binom{n-1}{l-1} + \binom{n-1}{l} \right\} x^l y^{n-l} + y^n$$

where the following identity hold,

$$\binom{n-1}{l-1} + \binom{n-1}{l} = \frac{(n-1)!}{(l-1)!(n-1-l)!} \left(\frac{1}{n-l} + \frac{1}{l} \right) = \binom{n}{l}.$$

So the result is proven. You might know the result as *Pascal's triangle*.

Identity: Pascal's triangle

- This is a simple way to compute the binomial coefficients.

	1						
	1	1					
	1	2	1				
n	1	3	3	1			
	1	4	6	4	1		
	1	5	10	10	5	1	
	1	6	15	20	15	6	1
							l

- $\binom{n}{l}$ (where $n \geq 0, l \geq 0$) corresponds to entry (n, l) of the array and satisfies $\binom{n}{l} = \binom{n-1}{l-1} + \binom{n-1}{l}$ for $l = 1, \dots, n-1$.

Identity: Combination formula

- We have established analytically $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
- Consider n objects and select one of them arbitrarily.
- The nb. of different groups of r objects that can be formed from n objects is $\binom{n}{r}$, it is also equal to the sum of the nb. of groups of r objects from n objects which include the arbitrarily selected object, given by $\binom{n-1}{r-1}$, and the nb. of groups of r objects from n objects which exclude the arbitrarily selected object, given by $\binom{n-1}{r}$.

Method 2: Proof of the binomial theorem by combinatorial

- Introduce artificial indexes on x, y and consider the product

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$$

- Expansion of this product leads to 2^n coefficients; e.g. for $n = 2$

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

and for $n = 3$

$$\begin{aligned}(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) = & x_1x_2x_3 + x_1y_2x_3 + y_1x_2x_3 + y_1y_2x_3 \\ & + x_1x_2y_3 + x_1y_2y_3 + y_1x_2y_3 + y_1y_2y_3.\end{aligned}$$

- Among the 2^n terms, we will have terms with k x_i 's and $(n - k)$ y_i 's. Each such term corresponds to the selection of a group of k x_i 's among n possible terms, there are $\binom{n}{k}$ such terms. Hence the result follows as $x_i = x, y_i = y$.

1.4 (Continue) The Binomial Theorem: Examples

- **Example:** How many *non-empty* subsets are there of a set of n elements?
- **Answer:** The total number of non-empty sets is the number of sets with k elements where $k = 1, \dots, n$

$$\sum_{k=1}^n \binom{n}{k} = \sum_{k=0}^n \binom{n}{k} - \underbrace{\binom{n}{0}}_{=1} = (1+1)^n - 1 = 2^n - 1.$$

- **Example** (Th. Ex. 13). Show that $\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$.
- **Answer.** We use the binomial theorem for $x = -1$ and $y = 1$

$$(-1+1)^n = 0 = \sum_{k=0}^n \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$

1.4 (Continue) The Binomial Theorem: Examples

- **Example** (Th. Ex. 12, question a). Establish

$\sum_{k=1}^n k \binom{n}{k} = n \cdot 2^{n-1}$ by considering a set of n persons and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.

- **Answer:** For a committee of size k , there are $\binom{n}{k}$ possible choices for selecting the persons in the committee and k choices for the chairperson so $k \binom{n}{k}$ possible choices $\Rightarrow$ total is $\sum_{k=1}^n k \binom{n}{k}$. Alternatively, there are n possible choices for the chairperson and we have $\binom{n-1}{k-1}$ other possible persons to put in the committee if it is of size $k = 1, \dots, n$. By summing over k

$$\sum_{k=1}^n \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = (1+1)^{n-1} = 2^{n-1} \Rightarrow \text{total is } n \cdot 2^{n-1}$$

1.5 Multinomial Coefficients

- We now want to divide a set of n items into r distinct groups of respective sizes $n_1, n_2, \dots, n_r$ where $n_1 + n_2 + \dots + n_r = n$. The number of possibilities is

$$\frac{n!}{n_1!n_2!\cdots n_r!} := \binom{n}{n_1, n_2, \dots, n_r} \quad \text{Multinomial coefficient}$$

This is the same as the number of permutations of n items with n_1 alike, n_2 alike etc.

- **Example** (*Pb. 25*): The game of bridge is played by 4 players, each of who is dealt 13 cards. How many bridge deals are possible?
- **Answer**: There are $4 \times 13 = 52$ cards so $52!$ possible permutations. However, like all card games, any permutation of the cards received by a given player are irrelevant (order does not matter). So there are

$$\frac{52!}{13!13!13!13!}$$

different possible deals.

1.5 Multinomial Coefficients: Examples

- **Example:** A small company has 9 employees. Everyday the company has 4 persons working during the day, 3 other persons working at night and 2 others not working. How many different divisions of the 9 employees in these 3 groups is possible?
- **Answer:** The answer is

$$\binom{9}{4, 3, 2} = 1260.$$

- **Example** (Pb. 28): 8 teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?
- **Answer.** Each teacher has 4 possible choices (outcome) so the answer to the first question is simply $4^8 = 65536$. If each school must receive 2 teachers then we have

$$\binom{8}{2, 2, 2, 2} = 2520.$$

1.5 Multinomial Coefficients: Examples

- **Example:** In order to organize a basketball tournament, 20 children at a playground divide themselves in four teams of 5 players. How many different divisions are possible?
- **Answer:** The answer is NOT

$$\binom{20}{5, 5, 5, 5}$$

because the order of the four teams is irrelevant. It would be exact if being in the team A would be considered different from being in the team D. Here we are only interested in the possible divisions, so as there are $4!$ permutations between team “labels” then the answer is

$$\binom{20}{5, 5, 5, 5} / 4! = \binom{20}{5, 5, 5, 5, 4}.$$

1.5 Multinomial Coefficients: Examples

- **Example:** In the 1st round of a knockout tournament involving $n = 2^m$ players, the n players are divided in $n/2$ pairs. Each pair plays a game. The losers of the games are eliminated while the winners go on to the next round and the process is repeated until only one single player remains. Assume $n = 8$. How many possible outcomes are there for the initial round?
- **Answer:** If there was an ordering of the pairs then the number of possible pairs for the initial round is given by

$$\binom{8}{2, 2, 2, 2} = \frac{8!}{2^4}.$$

As there is no ordering of the pairs, then there are $\frac{8!}{2^4}/4!$ pairs. For each of these pairs then there are 2 possible outcomes in the game and there are 4 games, so there are

$$\frac{8!}{2^4 4!} \times 2^4 = \frac{8!}{4!}.$$

1.5 Multinomial Coefficients: Examples

- Alternative way to establish the result: pick 4 winners among the 8 players, $\binom{8}{4}$ possibilities, and match them with the 4 losers, there are $4!$ ways to do this so $\binom{8}{4} 4! = \frac{8!}{4!}$.
- **Question:** How many outcomes of the tournaments are possible, where an outcome gives complete information for all rounds?
- **Answer:** Similarly, we have $\frac{4!}{2!}$ possible outcomes for the second round and $\frac{2!}{1!}$ for the third (and final) round. So by the principle of counting, there are

$$\frac{8!}{4!} \times \frac{4!}{2!} \times \frac{2!}{1!} = 8!$$

In the general case of $n = 2^m$, there are m rounds and $n!$ possible outcomes.

1.5 (Continue) The Multinomial Theorem

Theorem

We have

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, n_2, \dots, n_r): \\ n_1 + n_2 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \dots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

- **Example:** Compute $(x_1 + x_2 + x_3)^3$.
- **Answer:** We have

$$\begin{aligned}(x_1 + x_2 + x_3)^3 &= \binom{2}{2, 0, 0} x_1^2 x_2^0 x_3^0 + \binom{2}{0, 2, 0} x_1^0 x_2^2 x_3^0 + \\ &\binom{2}{0, 0, 2} x_1^0 x_2^0 x_3^2 + \binom{2}{1, 1, 0} x_1^1 x_2^1 x_3^0 + \binom{2}{1, 0, 1} x_1^1 x_2^0 x_3^1 \\ &+ \binom{2}{0, 1, 1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\end{aligned}$$

Proof of the multinomial theorem by Combinatorial

- Introduce artificial indexes on x_i and consider the product

$$(x_{1,1} + x_{2,1} + \cdots + x_{r,1}) (x_{1,2} + x_{2,2} + \cdots + x_{r,2}) \times \cdots \\ \cdots \times (x_{1,n} + x_{2,n} + \cdots + x_{r,n})$$

- Expansion of this product leads to r^n coefficients.
- Among the r^n terms, we will have terms with n_1 $x_{1,i}$'s, n_2 terms $x_{2,i}$'s, ..., n_r terms $x_{r,i}$'s. This corresponds to the selection of groups of n_1 terms, n_2 terms, ... n_r terms such that $n_1 + n_2 + \cdots + n_r = n$. Hence there $\binom{n}{n_1, n_2, \dots, n_r}$ such terms. The result follows as $x_{k,i} = x_k$ for $i = 1, \dots, n$.