

Chapter 4:

Allocating Resources Over Time

Objective

Explain the concept of compounding and discounting and to provide examples of real life applications

Introduction: Time Value of Money (TVM)

\$20 today is worth more than the expectation of \$20 tomorrow because:

- a bank would pay interest on the \$20**
- inflation makes tomorrow's \$20 less valuable than today's**
- uncertainty of receiving tomorrow's \$20**

Interest

- The increase in an amount of money as a result of interest earned
- Simple interest= amount of money (principal) * annual interest rate*time period
- Compound interest is interest earned on principal plus interest that was earned earlier

4.1 Compounding

- Assume that the interest rate is 10% p.a. (per annum)
- What this means is that if you invest \$1 for one year, you have been promised $\$1 * (1 + 10/100)$ or \$1.10 next year
- Investing \$1 for yet another year promises to produce $1.10 * (1 + 10/100)$ or \$1.21 in 2-years

Value of Investing \$1

- Continuing in this manner you will find that the following amounts will be earned:

1 Year	\$1.1
2 Years	\$1.21
3 Years	\$1.331
4 Years	\$1.4641

Value of \$5 Invested

- More generally, with an investment of \$5 at 10% we obtain

1 Year $\$5 \times (1 + 0.10)$ \$5.5

2 years $\$5.5 \times (1 + 0.10)$ \$6.05

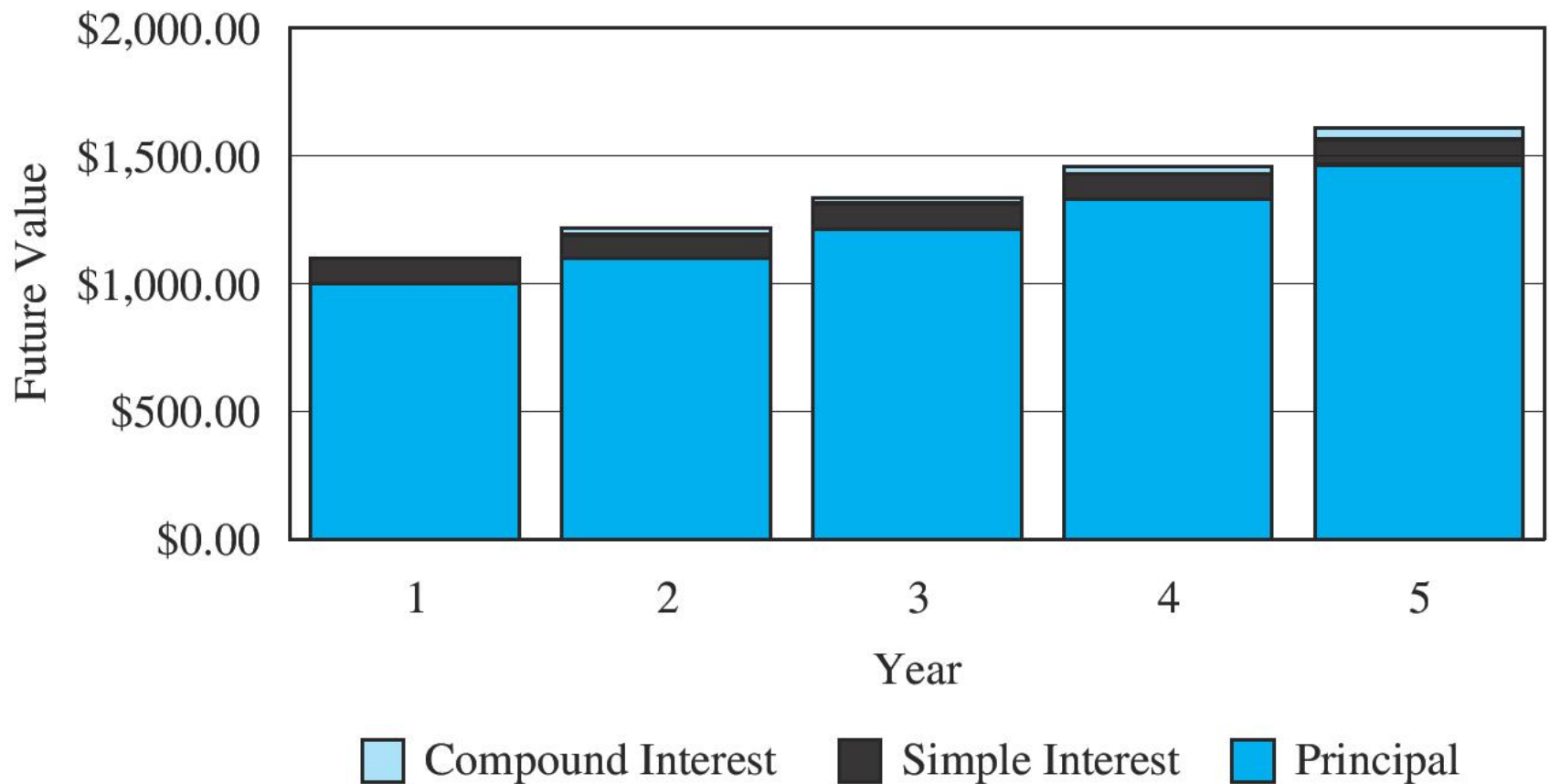
3 years $\$6.05 \times (1 + 0.10)$ \$6.655

4 Years $\$6.655 \times (1 + 0.10)$ \$7.3205

Generalizing the method

- Generalizing the method requires some definitions. Let
 - i be the (periodic) interest rate
 - n be the life of the lump sum investment (number of compounding periods)
 - PV be the present value
 - FV be the future value

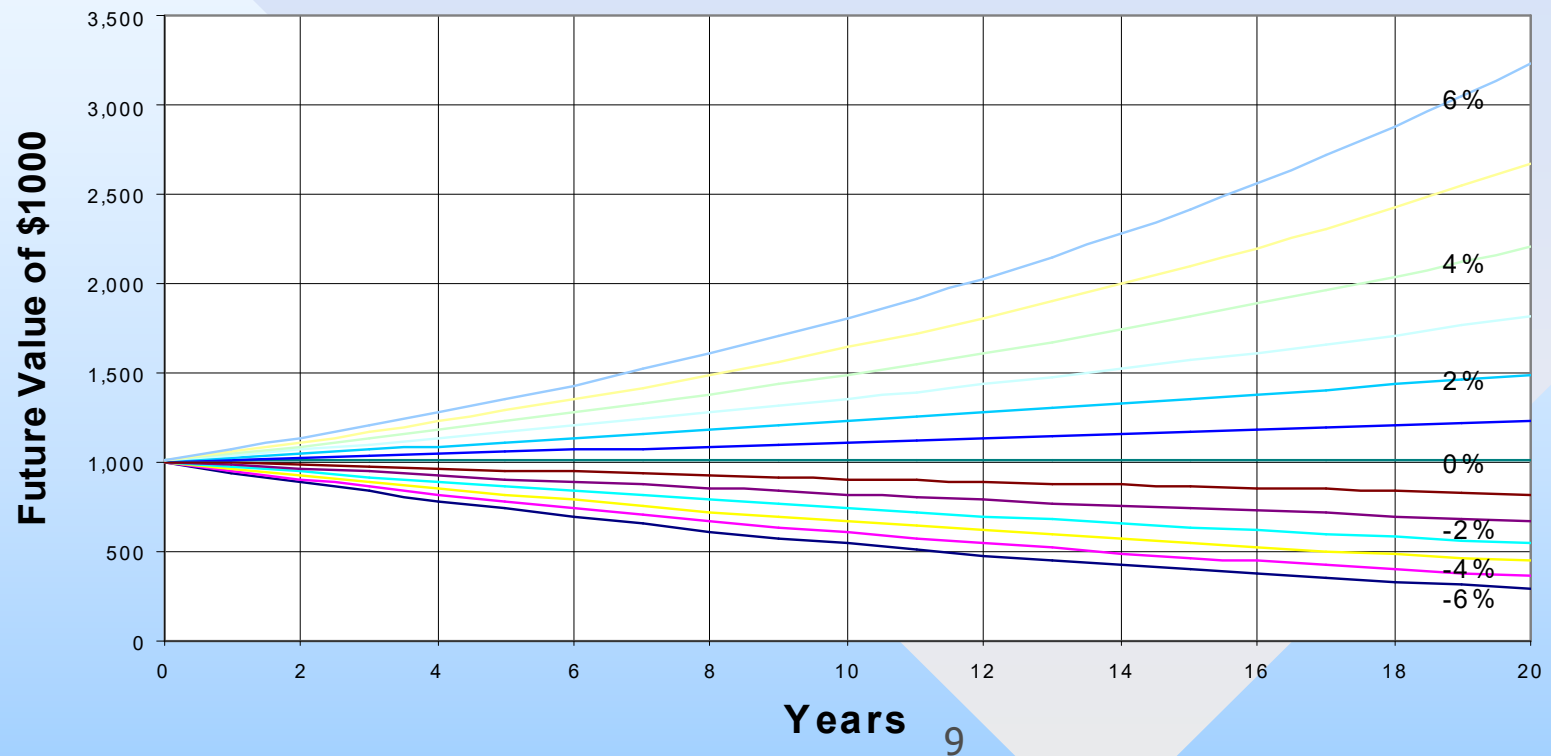
Future Value and Compound Interest



Future Value of a Lump Sum

$$FV = PV * (1 + i)^n$$

FV with growths from -6% to +6%



Example: Future Value of a Lump Sum

- Your bank offers a CD with an interest rate of 3% for a 5 year investment.

$$\begin{aligned}FV &= PV * (1 + i)^n \\&= \$1500 * (1 + 0.03)^5 \\&= \$1738.1111145\end{aligned}$$

- You wish to invest \$1,500 for 5 years, how much will your investment be worth?

n	5
i	3%
PV	1,500
FV	?
Result	1738.911111

RULE OF 72

- This rule says that the number of years it takes for a sum of money to double in value (“the doubling time”) is approximately equal to the number 72 divided by the interest rate expressed in percent per year
- Doubling Time =
$$\frac{72}{\text{Interest Rate}}$$

Hint: Rounding is common sense “plus” tradition

- It's important to round appropriately
 - In a \$ 7 billions project, rounding might be to the nearest \$'000,000
 - Your check book should be rounded to \$0.01
 - In an accounting situation, *any* unexpected error, however small could be the result of two larger compensating errors. Accordingly, they need to be resolved
- Avoid any truncation *within* a calculation

Present Value of a Lump Sum

$$FV = PV * (1 + i)^n$$

Divide both sides by $(1 + i)^n$ to obtain :

$$PV = \frac{FV}{(1 + i)^n} = FV * (1 + i)^{-n}$$

Example: Present Value of a Lump Sum

- You have been offered \$40,000 for your printing business, payable in 2 years. Given the risk, you require a rate of return of 8%. What is the present value of the offer?

$$\begin{aligned}PV &= \frac{FV}{(1+i)^n} \\&= \frac{40,000}{(1+0.08)^2} \\&= 34293.55281 \\&\cong \$34,293.55 \text{ today}\end{aligned}$$

Lump Sums Formulae

- You have solved a **present value** and a **future value** of a lump sum. There remains two other variables that may be solved for
 - Periodic interest rate, i
 - number of compounding periods, n

Solving Lump Sum Cash Flow for Interest Rate

$$FV = PV * (1 + i)^n$$

$$\frac{FV}{PV} = (1 + i)^n$$

$$(1 + i) = \sqrt[n]{\frac{FV}{PV}}$$

$$i = \sqrt[n]{\frac{FV}{PV}} - 1$$

Example: Interest Rate on a Lump Sum Investment

- If you invest \$15,000 for ten years, you receive \$30,000. What is your annual rate of return?

$$\begin{aligned}i &= \sqrt[n]{\frac{FV}{PV}} - 1 \\&= \sqrt[10]{\frac{30000}{15000}} - 1 = \sqrt[10]{2} - 1 = 2^{\frac{1}{10}} - 1 \\&= 0.071773463 \\&= 7.18\% \text{ (to the nearest basis point)}\end{aligned}$$

Review of Logarithms

- Logarithms are important in finance because growth is related to the exponential, and the exponential is the inverse function of the logarithm
- Logarithms may have different bases, but in finance we need only the *natural logarithm*, that is the logarithm of base *e*.
 - The *e* is the irrational number that may be approximated as 2.718281828. It is easy to remember because it starts to repeat, but don't be fooled, it doesn't, and it **is** *irrational*

Review of Logarithms

- The basic properties of logarithms that are used by finance are:

$$e^{\ln(x)} = x, \quad x > 0$$

$$\ln(e^x) = x$$

$$\ln(x * y) = \ln(x) + \ln(y)$$

$$\ln(x^y) = y \ln(x)$$

Review of Logarithms

- The following properties are easy to prove from the last ones, and are useful in finance

$$\ln(x / y) = \ln(x) - \ln(y)$$

$$\ln(x * y * z) = \ln(x) + \ln(y) + \ln(z)$$

$$\ln(x + y) \neq \ln(x) * \ln(y)$$

Solving Lump Sum Cash Flow for Number of Periods

$$FV = PV * (1 + i)^n$$

$$\frac{FV}{PV} = (1 + i)^n$$

$$\ln\left(\frac{FV}{PV}\right) = \ln\left((1 + i)^n\right) = n * \ln(1 + i)$$

$$n = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + i)} = \frac{\ln(FV) - \ln(PV)}{\ln(1 + i)}$$

4.2 The Frequency of Compounding

- Annually: 1 compounding period per year
- Semiannually: 2 compounding periods per year
- Quarterly: 4 compounding periods per year
- Monthly: 12 compounding periods per year
- Weekly: 52 compounding periods per year
- Daily: 365 compounding periods per year
- Continuously: infinite compounding periods per year

The Frequency of Compounding

- **APR:** annual percentage rate
- **Periodic interest rate**
= $\text{APR} / \text{compounding periods}$
- To make interest rate comparable, we define **effective annual rate** (EFF) as the equivalent interest rate if compounding were only once per year

The Frequency of Compounding

- Assume n compounding periods in a year and a nominal rate k per year compounded periodically. That is the periodic rate is k/n per period.
- Invest \$1 for one year to obtain $\$1 \cdot (1 + k/n)^n$, producing an effective annual rate over the year of $(\$1 \cdot (1 + k/n)^n - \$1) / \$1 = (1 + k/n)^n - 1$

Credit Card

- If the credit card pays an APR of 18% per year compounded monthly. The (periodic) monthly rate is $18\%/12 = 1.5\%$ so the effective annual rate is $(1+0.015)^{12} - 1 = 19.56\%$
- Two equal APRs with different frequencies of compounding have different effective annual rates:

Effective Annual Rates of an APR of 18%

Annual Percentage rate	Frequency of Compounding	Annual Effective Rate
18	1	18.00
18	2	18.81
18	4	19.25
18	12	19.56
18	52	19.68
18	365	19.7164245

The Frequency of Compounding

- Note that as the frequency of compounding increases, so does the annual effective rate
- What occurs as the frequency of compounding rises to infinity?

$$EFF = \lim_{m \rightarrow \infty} \left(\left(1 + \frac{k_m}{m} \right)^m \right) - 1 = e^{k_\infty} - 1$$

The Frequency of Compounding

- The effective annual rate that's equivalent to an annual percentage rate of 18% compounded continuously is then $e^{0.18} - 1 = 19.7217363\%$
- More precision shows that moving from daily compounding to continuous compounding gains 0.53 of one basis point. Why?

The Frequency of Compounding

- A bank determines that it needs an effective rate of 12% on car loans to medium risk borrowers
- What annual percentage rates may it offer?

The Frequency of Compounding

$$1 + EFF = \left(1 + \frac{k_m}{m}\right)^m$$

$$1 + \frac{k_m}{m} = (1 + EFF)^{\frac{1}{m}}$$

$$k_m = m * \left((1 + EFF)^{\frac{1}{m}} - 1\right)$$

The Frequency of Compounding

Annual Effective Rate	Compounding Frequency	Annual Percentage Rate
12	1	12.00
12	2	11.66
12	4	11.49
12	12	11.39
12	52	11.35
12	365	11.33
12	Infinity	11.33

The Frequency of Compounding

- Many lenders and borrowers do not have a clear understanding of APRs, but institutional lenders and borrowers do
- Institutions are therefore able to extract a few basis points from consumers, but why bother?

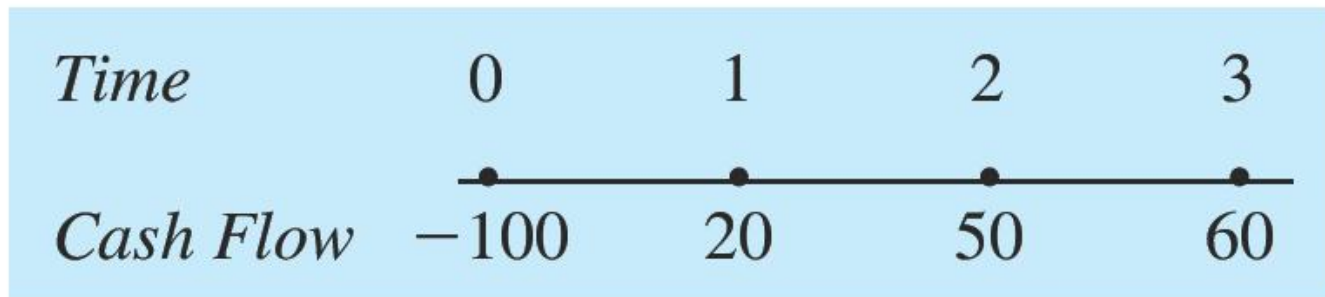
The Frequency of Compounding

- Financial intermediaries profit from differences in the lending and borrowing rates. Overheads, bad loans and competition results in a narrow margin. Small rate gains therefore result in a large increases in institutional profits
- In the long term, ill-informed consumers lose because of compounding

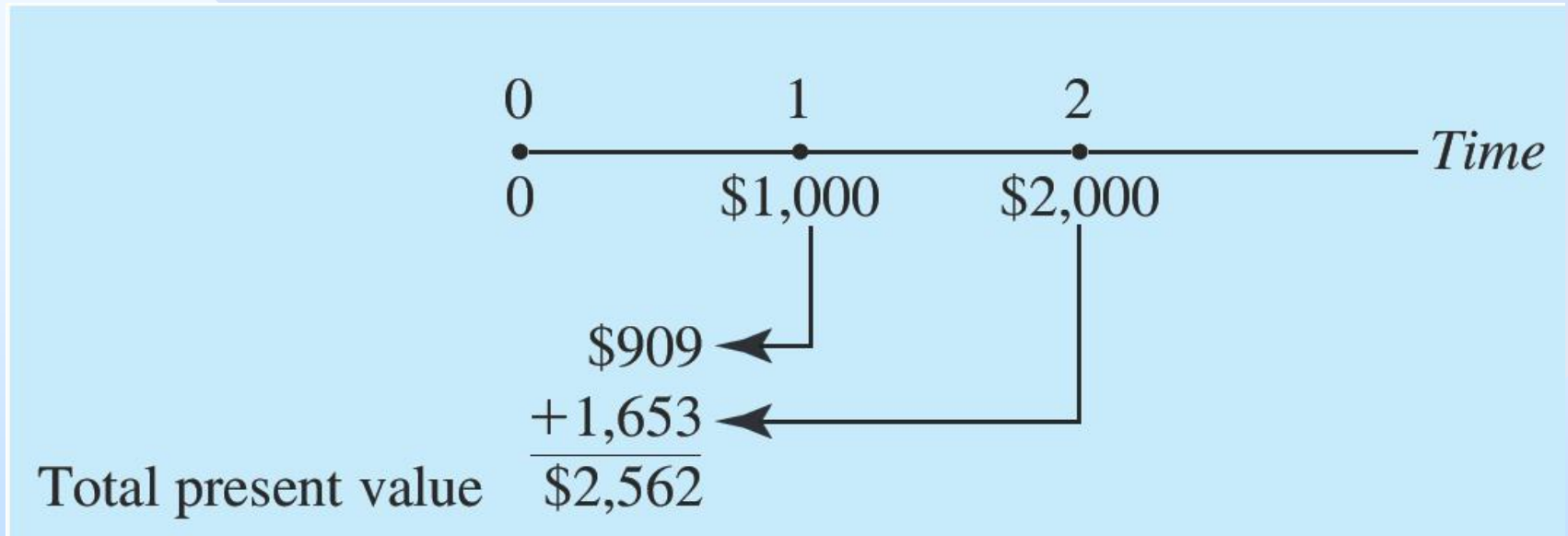
4.5 Multiple Cash Flows

- Time Lines
- Future Value of a Stream of Cash Flow
- Present Value of a Stream of Cash Flows
- Investing with Multiple Cash Flows

Time Line



Present Value of Multiple Cash Flows



Value of Cash flows

- Net Present Value (NPV) approach
 - Determine discount rate first
 - Computed Net present value
 - Decision rule:
 - accept If $NPV \geq 0$; Reject if $NPV < 0$
- Internal Rate of Return (IRR) approach
 - Compute IRR which makes $NPV=0$
 - Compare IRR with discount rate
 - Decision rule:
 - Accept if $IRR \geq \text{discount rate}$; Reject otherwise.

NPV and IRR

- NPV is the difference between the PV of all future cash inflows and the PV of all current and future cash outflows

$$NPV = \sum \{ CF_t / (1 + r)^t \}$$

- IRR is the discount rate that makes the NPV equals zero

$$0 = \sum \{ CF_t / (1 + IRR)^t \}$$

How to determine discount rate?

Appropriate discount rate is actually risk adjusted required rate of return

– Basic principle:

- higher (perceived) risk, higher discount rate

– Equilibrium interest rate

- Real risk-free rate + risk premiums

– Individual variation

- More risky project, higher discount rate
- More risk averse, higher discount rate

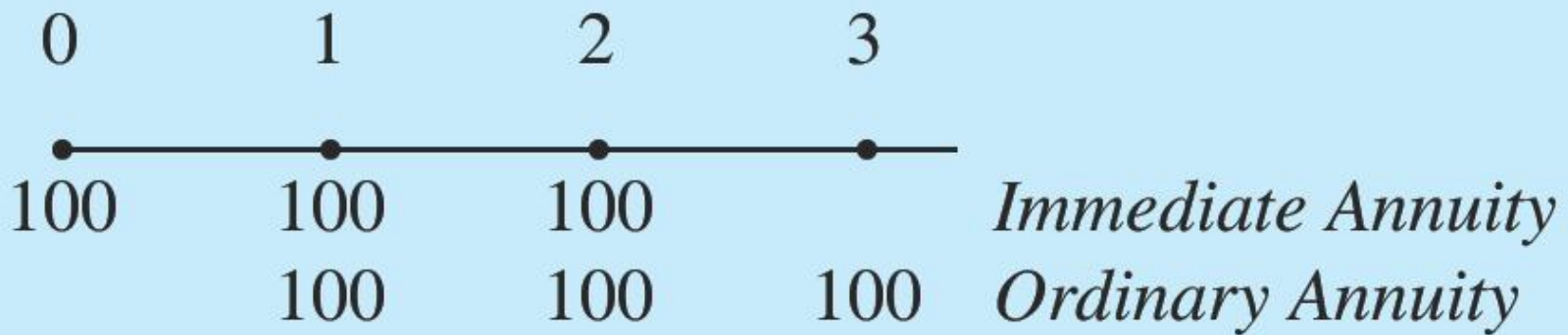
– Type of investors

- Lower discount rate for debt investors because they stands first to receive cash flow

4.6 Annuities

- Simple case of a stream of cash flow
 - A series of equal amounts for more than one period with identical amount each period
- Financial analysts use several annuities with differing assumptions about the first payment. We will examine just two:
 - *regular annuity* with its first coupon one period from now
 - *annuity due* with its first coupon today,

Cash Flow Diagram of Annuities



Rationale for Annuity Formula

- a sequence of equally spaced identical cash flows is a common occurrence, so automation pays off
- a typical annuity is a mortgage which may have 360 monthly payments, a lot of work for using elementary methods

Assumptions Regular Annuity

- the first cash flow will occur exactly one period from now
- all subsequent cash flows are separated by exactly one period
- all periods are of equal length
- the term structure of interest is flat
- all cash flows have the same (nominal) value
- the present value of a sum of present values is the sum of the present values

Annuity Formula Notation

- **PV** = the present value of the annuity
- **i** = interest rate to be earned over the life of the annuity
- **n** = the number of payments
- **pmt** = the periodic payment
- **FV** = the Future value of the annuity

FV and PV of an Annuity

$$FV_a = PMT \left[\frac{(1+i)^n - 1}{i} \right]$$

$$PV_a = PMT \left[\frac{(1+i)^n - 1}{i(1+i)^n} \right]$$

Derivation of PV of Annuity Formula: Algebra. 1 of 5

$$PV = \frac{pmt}{(1+i)^1} + \frac{pmt}{(1+i)^2} +$$
$$\frac{pmt}{(1+i)^3} + \dots + \frac{pmt}{(1+i)^{n-1}} + \frac{pmt}{(1+i)^n}$$

Derivation of PV of Annuity Formula: Algebra.

2 of 5

$$PV = pmt * \left\{ \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \dots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} \right\}$$

Derivation of PV of Annuity Formula: Algebra. 3 of 5

$$PV^*(1+i) = pmt^*(1+i) * \left\{ \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \dots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} \right\}$$

Derivation of PV of Annuity Formula: Algebra. 4 of 5

$$\begin{aligned}
 PV * (1+i) &= pmt * \left\{ \frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \right. \\
 &\quad \left. \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + \left[\frac{1}{(1+i)^n} - \frac{1}{(1+i)^n} \right] \right\} \\
 &= pmt * \frac{1}{(1+i)^0} + pmt * \left\{ \frac{1}{(1+i)^1} + \right. \\
 &\quad \left. \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} \right\} - pmt \frac{1}{(1+i)^n}
 \end{aligned}$$

Derivation of PV of Annuity Formula: Algebra. 5 of 5

$$PV * (1 + i) = pmt * \frac{1}{(1 + i)^0} + PV - pmt \frac{1}{(1 + i)^n}$$

$$PV * (1 + i) - PV = pmt - pmt \frac{1}{(1 + i)^n}$$

$$PV = \frac{pmt * \left\{1 - \frac{1}{(1 + i)^n}\right\}}{i} = \frac{pmt}{i} * \left(1 - \frac{1}{(1 + i)^n}\right)$$

PV of Annuity Formula

$$PV = \frac{pmt * \left\{ 1 - \frac{1}{(1+i)^n} \right\}}{i}$$
$$= \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n} \right)$$

PV Annuity Formula: Payment

$$PV = \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n} \right)$$
$$= \frac{pmt}{i} * \left(1 - (1+i)^{-n} \right)$$

$$pmt = \frac{PV * i}{\left(1 - (1+i)^{-n} \right)}$$

PV Annuity Formula: Number of Payments

$$PV = \frac{pmt}{i} * (1 - (1+i)^{-n}); \quad \frac{PV * i}{pmt} = 1 - (1+i)^{-n}$$

$$(1+i)^{-n} = 1 - \frac{PV * i}{pmt}; \quad -n * \ln(1+i) = \ln\left(1 - \frac{PV * i}{pmt}\right)$$

$$(1+i)^{-n} = 1 - \frac{PV * i}{pmt}; \quad n = -\frac{\ln\left(1 - \frac{PV * i}{pmt}\right)}{\ln(1+i)}$$

PV Annuity Formula: Return

- **There is no transcendental (closed-form or analytical) solution to the PV of an annuity equation in terms of the interest rate.**

4.7 Perpetual Annuities

- **Recall the annuity formula:**

$$PV = \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n} \right)$$

- Let $n \rightarrow \text{infinity}$ with $i > 0$:

$$PV = \frac{pmt}{i}$$

Growing Annuities

- They are often more appropriate in day-to-day situations than annuities
- PMT grow by g every year, then the Gordon model is the following:

$$PV = PMT_1 / (i - g)$$

where PMT_1 is the first payment

FV and PV of Annuity Due

- Annuity due is an annuity payment at the beginning of each period

$$FV_{ad} = PMT \left[\frac{(1+i)^n - 1}{i} \right] (1+i)$$

$$PV_{ad} = PMT \left[\frac{(1+i)^n - 1}{i(1+i)^n} \right] (1+i)$$

4.8 Loan Amortization (分期償付) : Mortgage

- early repayment permitted at any time during mortgage's 360 monthly payments
- market interest rates may fluctuate, but the loan's rate is a constant 1/2% per month
- the mortgage requires 10% equity and “three discount points” (A point=1% of the loan amount).
- assume a ¥1,000,000 house price

Mortgage: The payment

- We will examine this problem using a financial calculator
- The first quantity to determine is the amount of the loan and the points

$$\begin{aligned}\text{Loan} &= ¥ 1,000,000 * (1 - 0.1) \\ &= ¥ 900,000\end{aligned}$$

$$\begin{aligned}\text{Points} &= ¥ 1,000,000 * (1 - 0.1) * 0.03 \\ &= ¥ 27,000\end{aligned}$$

- The borrower only gets 900,000-27,000

Calculator Solution

Online tool: www.mortgagecalculator.org

n	i	PV	FV	PMT	Result
360	0.5%	900,000	0	?	-5,395.96

This is
the monthly
repayment

PMT calculation

$$\text{PMT} = \text{AMT. BORROWED} (\text{MC}_{i,n})$$

where MC stands for mortgage constant, the payment per dollar borrowed at a given (periodic) interest rate, i , and the number of periods, n .

$$\text{PMT} = \$900,000 (\text{MC}_{0.5\%,360})$$

$$\text{PMT} = \$900,000 \times$$

$$\frac{(0.5\%) (1+0.5\%)^{360}}{(1+0.5\%)^{360} - 1}$$

$$= \$ 5,395.96$$

Loan Amortization (分期偿付)

- The process of paying off a loan's principal gradually over the term
- The first repayment is 5,395.96 of which interest portion is $900,000 \times 0.5\% = 4500$
- Each month's interest is calculated as the outstanding balance at the beginning of the month times the monthly interest rate
- Principal Portion is $5395.96 - 4500 = 895.96$
- The outstanding balance after 1 month is $900000 - 895.96 = 889,104.04$

Mortgage: Early Repayment

- Assume that the family plans to sell the house after exactly 60 payments, what will be the outstanding principal?
- \$900,000-principal portion of the 60 monthly payments
- $\text{PMT} / (\text{MC}_{0.5\%, 300}) = ?$

Mortgage Repayment: Issues

- The outstanding principal is the present value (at repayment date) of the remaining payments on the mortgage
- There are in this case $360 - 60 = 300$ remaining payments, starting with the one 1-month from now

Calculator Solution

n	i	PV	FV	PMT	Result
360	.5%	900,000	0	?	-5,395.96
300	.5%	?	0	-5,395.96	837,490

Outstanding @ 60 Months

Summary of Payments

- The family has made 60 payments =
 $\text{¥ } 5,395.96 * 12 * 5 = \$323,757.28$
- Their mortgage principal repayment =
 $900,000 - 837,489.22 = \text{¥ } 62,510.78$
- Interest = payments - principal reduction
 $= 323,757.28 - 62,510.78 = \text{¥ } 261,246.50$

Avoid Adding Cash Flows From Different Periods

- In the above slide, we broke one of the cardinal rules of finance: We bundled the cash flows for 5-years by adding them together
- This kind of analysis can lead to inappropriate financial decisions, such as early repayment of a mortgage

A Result of Breaking the Rule

- Given the tax advantages of a mortgage, and the fact it is collateralized, their interest rates are quite low
- Some financial pundits recommend adding (say 10%) to monthly payments to reduce the mortgage life by 5- to 10-years
- At your age, investing that extra 10% in a mutual fund may be more appropriate

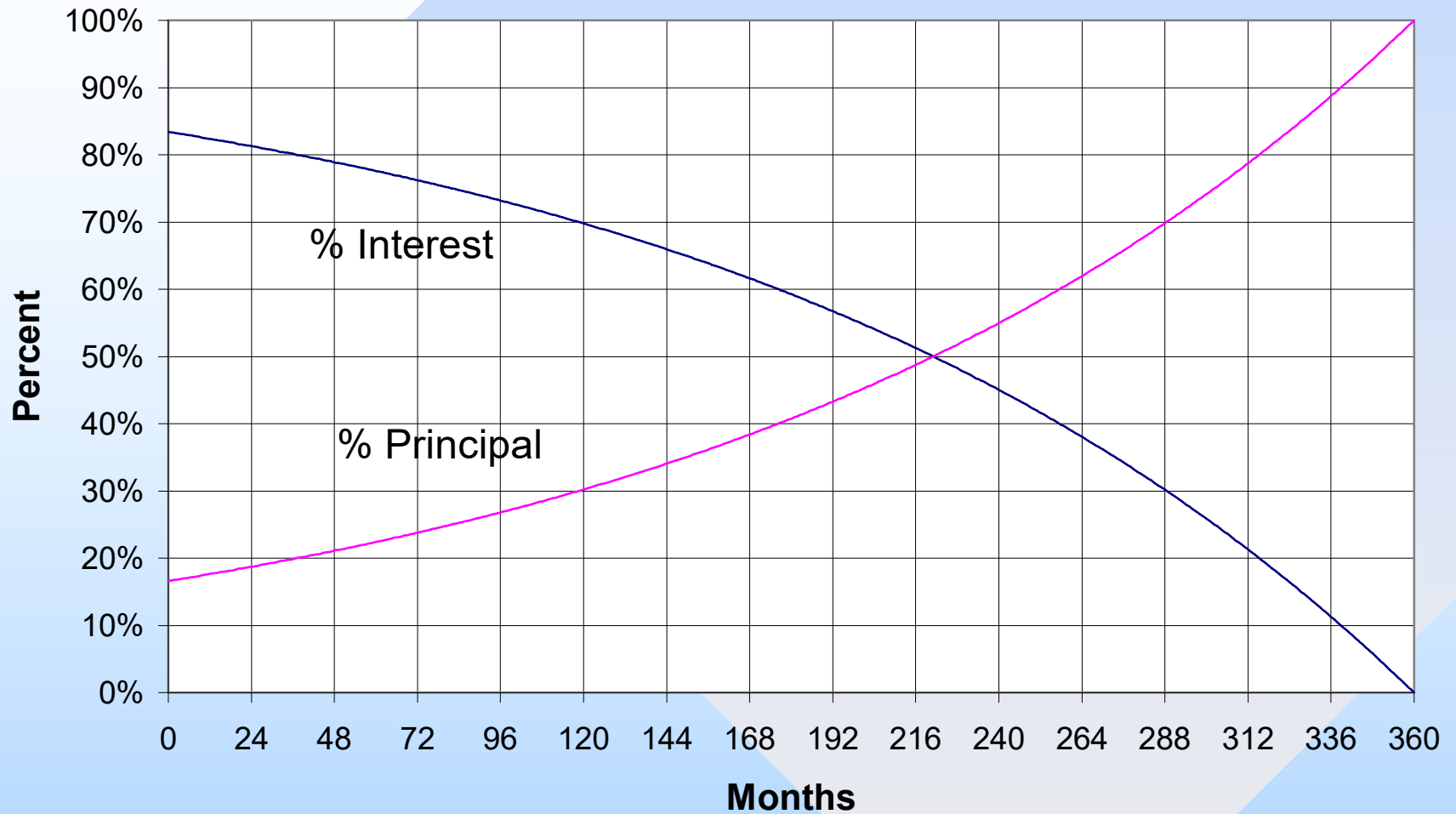
A Result of Breaking the Rule

- The pundits make their argument by adding (without discounting!) the difference in the cash flows between the scenarios. This is typically a huge sum of money, and this is what is “saved”
- When discounted appropriately, there are no significant savings. There are huge opportunity losses for those willing to accept the risk of a stock mutual fund

Outstanding Balance as a Function of Time

- The following graphs illustrate that in the early years, monthly payment are mostly interest. In latter years, the payments are mostly principal
- Only the interest portion is tax-deductible, so the tax shelter decays

Percent of Interest and Principal



4.9 Exchange Rates and Time Value of Money

Financial Decision in an International Context

- International currency investors borrow and lend in
 - Their own currency
 - The currency of countries with which they do business but wish to hedge
 - Currencies that appear to offer a better deal
- Exchange rate fluctuations can result in unexpected gains and losses

Computing NPV in Different Currencies

- In any time-value-of-money calculation, the cash flows and interest rates must be denominated in the same currency
 - USA project U requires an investment of \$10,000, as does a Japanese project J. U generates \$6,000/year for 5 years, and project J generates ¥575,000/year for 5 years
 - The US interest is 6%, the Japanese interest is 4%, and the current exchange rate is \$0.01/ ¥

Solution

- Using your financial calculator
 - Determine the present value of U in \$ by discounting the 5 payments at 6%, and subtract the initial investment of \$10,000
 - Determine the present value of J in ¥ by discounting the 5 payments at 4%, and subtract the initial investment of ¥ 1,000,000
 - Obtain \$15,274 & ¥1,5599,798 respectively

Solution

- Convert the ¥1,5599,798 to \$ using the current exchange rate to obtain \$15,600
- The Japanese NPV of ¥ of \$15,600 is higher than the USA NPV or \$15,274, so invest in the Japanese project assuming no exchange rate risk

4.10 Inflation and Discounted Cash Flow Analysis

- We will use the notation
 - i_n the rate of interest in nominal terms
 - i_r the rate of interest in real terms
 - r the rate of inflation
- From chapter 2 we have the relationship

$$1 + i_r = \frac{1 + i_n}{1 + r} \Leftrightarrow i_r = \frac{i_n - r}{1 + r}$$

Illustration

- What is the real rate of interest if the nominal rate is 8% and inflation is 5%?

$$1 + i_r = \frac{1 + i_n}{1 + r} \Leftrightarrow i_r = \frac{i_n - r}{1 + r}$$

$$i_r = \frac{0.08 - 0.05}{1.05} = 0.0286 = 2.86\%$$

- *The real rate or return determines the spending power of your savings*
- *The nominal value of your wealth is only as important as its purchasing power*

Investing in Inflation-protected CD's

- You have decided to invest \$10,000 for the next 12-months. You are offered two choices
 - A nominal CD paying a 8% return
 - A real CD paying 3% + inflation rate
- If you anticipate the inflation being
 - Below 5% invest in the nominal security
 - Above 5% invest in the real security
 - Equal to 5% invest in either

Why Debtors Gain From Unanticipated Inflation

- You borrow \$10,000 at 8% interest. The today's spending power of the repayment is $\$10,000 * 1.08 / (1 + \text{inflation})$
 - If the actual inflation is the expected 6%, then the real cost of the loan in today's money is \$10,188.68
 - If the actual inflation is 10%, then the loan's real cost (in today's values) is \$9,818.18
- Unexpected inflation benefits borrower

Inflation and Present Value

- A common planning situation is determining how long it takes to save for something
- The problem is that the thing being saved for increases in (nominal) price due to inflation
- Never use a nominal interest rate when discounting real cash flow or a real interest rate when discounting nominal cash flows

Inflation and Present Value

- Illustration example in the textbook (P137):

If you convert from oil to gas heating, cost will reduce by \$500 for many years. And the installment cost is \$10,000

→ $IRR = 500 / 10,000 = 5\%$ in real term

Alternatively, deposit in a bank with 8% p.a.

→ Nominal rate is 8%

4.11 Taxes and Investment Decisions

- Rule:
 - Invest so as to maximize your after-tax rate of return, which is $(1 - \text{Tax rate})^*$ before-tax rate of return
- This is not at all the same thing as
 - Minimize the tax you pay (False)

Investing in Tax-Exempt Bonds

- In the USA, municipal bonds are exempt from income taxes
 - For example, it pays 6% per year
- Under what circumstances would you be indifferent to investing in an identical bond that pays tax if your marginal rate of tax is (say) 20%?